Worksheet on Homomorphisms in GAP

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Exercise 1 (Form some homomorphisms to quotients and compare their images)

a) *Let* `G:=SmallGroup(60,10)`. *Construct the homomorphism* $\varphi\colon G \to G/G'$.

b) *Let* $G = S_5$. *Construct the homomorphism* $\varphi\colon G \to G/G'$.

c) *Let* $G = SL_2(5)$. *Construct the homomorphism* $\varphi\colon G \to G/G'$.

d) *Can you explain the kind of image group, and why the third homomorphism looks so different.*

□

Exercise 2 (A homomorphism given by a rule) *Let* G *be the stabilizer in* $GL_4(3)$ *of the subspace spanned by the first two basis vectors:*

```
gap> sub:=IdentityMat(4,GF(3)){[1,2]};
[ [ Z(3)^0,0*Z(3),0*Z(3),0*Z(3) ], [ 0*Z(3),Z(3)^0,0*Z(3),0*Z(3) ] ]
gap> G:=Stabilizer(GL(4,3),sub,OnSubspacesByCanonicalBasis);
<matrix group of size 186624 with 7 generators>
```

Construct a homomorphism on $GL_2(3)$ *given by the action on the factor space.* □

Exercise 3 (Create a presentation and transfer to another free group)
Construct $S_4 = \langle e := (1,2,3,4), f := (1,2)\rangle$ *as a finitely presented group on these two generators (named* `e` *and* `f`*).*
Hint: *Use* `IsomorphismFpGroupByGenerators` *to get a presentation, then make a new free group with the generators named as you want and transfer the relators to the new group. Use* `FreeGroupOfFpGroup` *and* `RelatorsOfFpGroup` *to translate relators to the free group with the desired generator names.* □

Exercise 4 (Stabilizer of a set of sets by changing permutation representation)
The Fano plane *is a combinatorial configuration consisting of 7 points and 7 three-element subsets of these points:*

$$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$

The automorphism group of this configuration is the subgroup of S_7 that fixes these sets. To find it:

a) *Construct all 35 3-element sets out of 7 points and construct the homomorphism for the action of S_7 on these sets.*

b) *Represent the fano plane by a 7-of-35 set of points and calculate the stabilizer of this set under the image of the action in a)*

c) *Take the pre-image of this stabilizer.*

□

Exercise 5 (Automorphisms induced by normalizer in S_n)
Let $G = \langle(1,2),(3,4)(5,6),(3,6)(4,5),(7,8),(3,4)(9,10),(3,9)(4,10)\rangle$. What is the number of automorphisms of G that are induced by $N_{S_{10}}(G)$?
Hint: *Calculate the normalizer and use* `ConjugatorAutomorphism` *on its generators. Bonus: Can you find the same result from calculating the automorphism group* without *calculating the normalizer?* □

Exercise 6 (Outer automorphism of S_6) *Construct an outer automorphism of S_6.* □

Exercise 7 (Smallest faithful permutation degree) *Let* `G:=SmallGroup(48,48);`*. What is the smallest degree n, such that G is isomorphic to a subgroup of S_n?* □

Exercise 8 (Construct an (external) semidirect product) *Construct S_4 as semidirect product $S_3 \ltimes C_2^2$* □

Exercise 9 (Decompose a group formally as (internal) semidirect product)
Let `G:=PerfectGroup(IsPermGroup,960,1);` *and N its solvable radical (the largest solvable normal subgroup,* `RadicalGroup`*). Find a complement C to N in G, using the operation* `ComplementClassesRepresentatives`*. Form (directly, without using* `IsomorphismGroups`*) an isomorphism between G/N and C and use it to write elements of G as pairs in $C \ltimes N$. Build an isomorphism between G and an abstract semidirect product of N with G.* □

Exercise 10 (Quotients of a particular type — generating direct products)
Determine the largest k, such that A_6^k can be generated by 2 elements. Construct such a 2-element generating set. **Hint:**

a) `GQuotients` *finds Quotients of a given isomorphism group. What happens if you apply it to the free group?*

b) *What happens if you intersect the kernels of multiple quotients with the same simple image?*

c) *Combine the homomorphisms to one homomorphism into the direct product of the images.* □

Exercise 11 (Forming larger and larger quotients of finitely presented groups) *Let* $G = \langle x, y, u, v \mid x^2, y^3, u^{10}v^2uvuv^2, xy/u, xy^{-1}/v \rangle$. *Construct a large quotient of* G.
Hint: G *has order* $2^{21}3^4 5^2 13^2$.

a) *Use* `IsomorphismSimplifiedFpGroup` *to eliminate two generators.*

b) *Find subgroups of low index (as far as plausible, at least 30) and intersect. Take the homomorphism* φ *that is used to define this intersection.*

c) *Find a subgroup of small index (*`MaximalSubgroupClassReps`, `LowLayerSubgroups`*) of the image of* φ *such that its abelianization is different than the one of its pre-image under* φ.

d) *Repeat, if needed.*

□

Exercise 12 (Constructing and inducing representations) *Construct the* reduced permutation action *of* A_5 *over* $GF(3)$ *(That is the action on the vectors whose coefficients sum to one.) Then induce this representation to* $PSL_2(9)$.
Hint: *You need to construct the map*

$$g \mapsto \left(g^{\psi}; \widetilde{g_{1^{g^{-1}}}}^{\varphi}, \ldots, \widetilde{g_{n^{g^{-1}}}}^{\varphi}\right)$$

where $\widetilde{g_j} \in T$ *is defined by by* $r_j g = \widetilde{g_j} r_{j^g}$ *for coset representatives* r_j. □