Majorana and axial algebras

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Background

Background - the Monster group

- Denoted $\mathbb{M}$, the Monster group is the largest of the 26 sporadic groups in the classification of finite simple groups
- It was constructed by R. Griess in 1982 as $\mathrm{Aut}(V_{\mathbb{M}})$ where $V_{\mathbb{M}}$ is a 196 884 - dimensional, real, commutative, non-associative algebra known as the Griess or Monster algebra
- The Monster group contains two conjugacy classes of involutions - denoted 2A and 2B - and $\mathbb{M} = \langle 2A \rangle$
- If $t, s \in 2A$ then ts is of order at most 6 and belongs to one of nine conjugacy classes:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A.$$

Background - the Griess algebra

- In 1984, J. Conway showed that there exists a bijection ψ between the 2A involutions and certain idempotents in the Griess algebra called 2A-axes
- The 2A-axes generate the Griess algebra i.e. $V_{\mathbb{M}} = \langle\langle \psi(t) \, : \, t \in 2A \rangle\rangle$
- If $t, s \in 2A$ then the algebra $\langle\langle \psi(t), \psi(s) \rangle\rangle$ is called a dihedral subalgebra of $V_{\mathbb{M}}$ and has one of nine isomorphism types, depending on the conjugacy class of ts.

Background - the Griess algebra

Example

Suppose that $t, s \in 2A$ such that $ts \in 2A$ as well. Then the algebra

$$V := \langle\langle \psi(t), \psi(s) \rangle\rangle$$

is called the $2A$ dihedral algebra.

The algebra V also contains the axis $\psi(ts)$. In fact, it is of dimension 3:

$$V = \langle \psi(t), \psi(s), \psi(ts) \rangle_{\mathbb{R}}.$$

Background - the Majorana fusion law

The $2A$-axes of the Griess algebra are semisimple with eigenvalues 1, 0, $\frac{1}{4}$ and $\frac{1}{32}$.

That is to say, if $a \in V_{\mathbb{M}}$ is a $2A$-axis then

$$V_{\mathbb{M}} = V_1^a \oplus V_0^a \oplus V_{\frac{1}{4}}^a \oplus V_{\frac{1}{32}}^a$$

where

$$V_\mu^a = \{v \in V_{\mathbb{M}} \mid av = \mu v\}.$$

Background - the Majorana fusion law

Moreover, the $2A$-axes obey the Majorana fusion law. That is to say, if $a \in V_{\mathbb{M}}$ is a $2A$-axis, then

$$u \in V^a_\mu, v \in V^a_\nu \Rightarrow uv \in \bigoplus_{\lambda \in \mu * \nu} V^a_\lambda.$$

Where $\mu * \nu$ is a set given by the following table.

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Background - Jordan algebras

A Jordan algebra V is a non-associative, commutative algebra over a field such that for all $u, v \in V$, $(uv)(uu) = u(v(uu))$.

If a is an idempotent in a Jordan algebra then a is semisimple with eigenvalues 1, 0 and $\frac{1}{2}$.

If $u \in V_\mu^a$ and $v \in V_\nu^a$ then $uv \in \bigoplus_{\lambda \in \mu * \nu} V_\lambda^a$ where $\mu * \nu$ is a set given by the following table.

$*$	1	0	$\frac{1}{2}$
1	1	$\emptyset$	$\frac{1}{2}$
0	$\emptyset$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

Axial algebras

Axial algebras - fusion laws

Let R be a commutative ring with unity and let V be a commutative R-algebra.

Definition 2.1

A fusion law is a pair $(\mathcal{F}, *)$ such that $\mathcal{F} \subseteq R$ and such that $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ is a symmetric map.

Definition 2.2

If $a \in V$ is an idempotent then a is a $(\mathcal{F}, *)$-axis if

(i) a is semisimple and $V = \bigoplus_{\lambda \in \mathcal{F}} V_\lambda^a$;

(ii) for all $\mu, \nu \in \mathcal{F}$ we have $V_\mu^a V_\nu^a \subseteq \bigoplus_{\lambda \in \nu * \mu} V_\lambda^a$.

Axial algebras - fusion laws

Example

The Majorana fusion law, $(\mathcal{M}, *)$:

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Axial algebras - fusion laws

Example

The fusion law of Jordan type η :

$*$	1	0	η
1	1	$\emptyset$	η
0	$\emptyset$	0	η
η	η	η	$1, 0$

Axial algebras - fusion laws

Definition 2.3

A $(\mathcal{F}, *)$-axial algebra is a pair (V, A) where V is a commutative R-algebra and $A \subseteq V$ is a generating set of $(\mathcal{F}, *)$-axes.

Definition 2.4

A $(\mathcal{F}, *)$-axis is primitive if its 1-eigenspace is one dimensional, i.e. $V_1^{(a)} = \langle a \rangle_R$.

Definition 2.5

A $(\mathcal{F}, *)$-axial algebra (V, A) is primitive if a is primitive for all $a \in A$.

Axial algebras - Frobenius forms

Definition 2.6

A Frobenius form on a commutative algebra V is a (non-zero) bilinear form $\langle\ ,\ \rangle$ such that for all $u, v, w \in V$

$$\langle u, vw \rangle = \langle uv, w \rangle.$$

We usually require that a Frobenius form $\langle\ ,\ \rangle$ on an axial algebra (V, A) satisfies $\langle a, a \rangle = 1$ for all axes $a \in V$.

Majorana algebras

Majorana algebras

A Majorana algebra is a primitive axial algebra (V, A) that satisfies the Majorana fusion law and that admits a a Frobenius form $\langle \, , \, \rangle$ such that

- $\langle \, , \, \rangle$ is an inner product, i.e. for all $v \in V$,

$$\langle v, v \rangle = 0 \Leftrightarrow v = 0;$$

- $\langle \, , \, \rangle$ obeys Norton's inequality, i.e for all $u, v \in V$

$$\langle uu, vv \rangle \geq \langle uv, uv \rangle.$$

In this case, we call the elements of A Majorana axes.

Majorana algebras - Majorana involutions

Let (V, A) be an axial algebra that obeys the Majorana fusion law.

For $a \in A$, let $\tau(a) \in \mathrm{GL}(V)$ be such that

$$\tau(a) : v \mapsto \begin{cases} v & \text{if } v \in V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{4}}^{(a)} \\ -v & \text{if } v \in V_{\frac{1}{32}}^{(a)}. \end{cases}$$

We call the $\tau(a)$ the Majorana involutions of V.

Majorana algebras - Majorana involutions

Let (V, A) be an axial algebra that obeys the Majorana fusion law.

Proposition 3.1

For all $a \in A$, $\tau(a) \in \mathrm{Aut}(V)$, i.e. for all $u, v \in V$

$$u^{\tau(a)} v^{\tau(a)} = (uv)^{\tau(a)}.$$

Proposition 3.2

If V admits a Frobenius form then for all $a \in A$, $\tau(a)$ preserves the form, i.e. for all $u, v \in V$,

$$\langle u^{\tau(a)}, v^{\tau(a)} \rangle = \langle u, v \rangle.$$

Majorana algebras - the Griess algebra

Example

The Griess algebra $V_{\mathbb{M}}$ is a Majorana algebra such that

- the Majorana axes are the $2A$-axes;
- the Majorana involutions are the $2A$-involutions of $\mathbb{M}$.

Majorana algebras - Majorana representations

Definition 3.3

A Majorana representation is a tuple (G, T, V, φ, ψ) where

- G is a finite group;
- T is a set of involutions such that $G = \langle T \rangle$ and $T^G = T$;
- (V, A) is a Majorana algebra;
- $\varphi : G \to \mathrm{GL}(V)$ is a linear representation and $\psi : T \to A$ is a bijective map such that for all $t \in T$ and $g \in G$

$$\tau(\psi(t)) = \varphi(t) \text{ and } \psi(t^g) = \psi(t)^{\varphi(g)}.$$

Informally:

1. take a group G and a generating set of involutions T;
2. take a set of Majorana axes that are indexed by the elements of T, i.e. $A = \{a_t \mid t \in T\}$;
3. then G will act on A via its conjugation action on T;
4. construct the Majorana representation V generated by A and extend the action of G to the whole of V.

Majorana algebras - Sakuma's theorem

Theorem 3.4

Suppose that (V, A) is a Majorana algebra. Let $a, b \in A$ and let $U = \langle\langle a, b \rangle\rangle$. Then U has one of nine isomorphism types, each of which occurs as a subalgebra of the Griess algebra.

These algebras are called dihedral algebras and have dimension as given below.

type	$1A$	$2A$	$2B$	$3A$	$3C$	$4A$	$4B$	$5A$	$6A$
dim	1	3	2	4	3	5	5	6	8

Majorana algebras - Sakuma's theorem

Corollary 3.5

Let $U = \langle\langle a, b\rangle\rangle$ be a dihedral algebra of type NX then $\tau(a)$ and $\tau(b)$ generate in $\mathrm{GL}(V)$ a dihedral group D of order $2N$ that acts on U with kernel $Z(D)$.

In particular, if (G, T, V) is a Majorana representation then (G, T) is a 6-transposition group.

Definition 3.6

A pair (G, T) is a 6-transposition group if G is a group and T is a generating set of involutions of G such that $T^G = T$ and such that for all $t, s \in T$, $|ts| \leq 6$.

Majorana algebras - Sakuma's theorem

Corollary 3.7

The following are the only non-trivial embeddings of one dihedral algebra into another:

$$2A \hookrightarrow 4B, \qquad 2B \hookrightarrow 4A, \qquad 2A \hookrightarrow 6A, \qquad 3A \hookrightarrow 6A$$

The algorithm

The algorithm

Let (V, A) be an axial algebra. Then we say that V is k-closed if

$$V = \langle x_1 x_2 \dots x_k \mid x_i \in A \rangle.$$

In 2012, Ákos Seress announced an algorithm in GAP to construct 2-closed Majorana representations.

Sadly, he passed away shortly afterwards and the full details of his code were never recovered.

Markus Pfeiffer and I have reimplemented his algorithm and extended it to n-closed Majorana representations.

The algorithm

Consequences:

- construction of all previously known Majorana representations, including representations of A_7 and M_{11};
- classification-type results, e.g. minimal 3-generated Majorana algebras
- significant new examples including an infinite family of Majorana algebras.

The algorithm

Input: A finite group G and set of involutions T such that $G = \langle T \rangle$ and $T^G = T$.

Output: A Majorana algebra. In particular, a spanning set C of V along with the algebra and inner product values on $C \times C$.

Step 0 - shapes

Recall: we take $A = \{a_t \mid t \in T\}$ and $V = \langle\langle A \rangle\rangle$.

For all $t, s \in T$, if $|ts| = N$ then $\langle\langle a_t, a_s \rangle\rangle$ is a dihedral algebra of type NX for $X \in \{A, B, C\}$ (Theorem 3.4).

The "inclusions" of dihedral algebras (Corollary 3.7) put further restrictions on the possible types of the dihedral algebras.

If $\{t_0, s_0\}, \{t_1, s_1\}, \dots, \{t_k, s_k\}$ are the representatives of the orbits of G on unordered pairs of T then the shape of a representation (G, T, V) is a tuple consisting of the types of the algebras $\langle\langle a_{t_i}, a_{s_i} \rangle\rangle$ for $1 \leq i \leq k$.

Step 0 - shapes

Example

Let $G = S_4$ and $T = (1,2)^G$. There are three orbits:

$\{t, s\}$	$\langle\langle a_t, a_s \rangle\rangle$
$\{(1,2),(1,2)\}$	$1A$
$\{(1,2),(1,3)\}$	$3A$ or $3C$
$\{(1,2),(3,4)\}$	$2A$ or $2B$

So there are four possible shapes of a Majorana representation (G, T, V).

Step 1 - setup

We start by attempting to construct the 2-closed algebra, i.e. $\langle a_t a_s \mid t, s \in T \rangle$.

The dihedral algebras $1A$, $2B$, $2A$, $3C$ and $4B$ are 1-closed but the dihedral algebras $3A$, $4A$, $5A$ and $6A$ are 2-closed.

Record any new spanning set vectors coming from the dihedral algebras, along with the Majorana axes, in a list called `coords`.

Also record all algebra and inner products and eigenvectors coming from the known structures of the dihedral algebras.

Step 1 - setup

The action of G preserves the algebra and inner product on V.

We calculate representatives of the orbits of G on unordered pairs of elements of `coords` and store them in a list called `pairreps`.

We only store the algebra and inner product values for the pairs of vectors given by `pairreps`.

We also store other data that allows us to recover the product of two generic vectors from these representatives.

The group G acts on `coords` via signed permutations.

Step 1 - setup

Example

Take $G = S_4$, $T = (1,2)^G$, shape $= (1A, 3A, 2B)$. Then V contains four $3A$ dihedral algebras:

$$
\begin{aligned}
U_1 &:= \langle\langle a_{(2,3)}, a_{(2,4)} \rangle\rangle \\
U_2 &:= \langle\langle a_{(1,3)}, a_{(1,4)} \rangle\rangle \\
U_3 &:= \langle\langle a_{(1,2)}, a_{(1,4)} \rangle\rangle \\
U_4 &:= \langle\langle a_{(1,2)}, a_{(1,3)} \rangle\rangle.
\end{aligned}
$$

So `coords` $= \{a_t \mid t \in T\} \cup \{u_i \mid 1 \leq i \leq 4\}$ where the u_i are called $3A$-axes and $u_i \in U_i$ for $1 \leq i \leq 4$.

Step 2 - inner products

Using the fact that $\forall u, v, w \in V$,

$$\langle uv, w \rangle = \langle u, vw \rangle$$

we can find new inner product values.

Step 2 - inner products

Example

$$G = S_4,\ T = (1,2)^G,\ \texttt{coords} = \{a_t \mid t \in T\} \cup \{u_i \mid 1 \leq i \leq 4\}.$$

From the known values of dihedral algebras, we know that

$$a_{(1,3)}a_{(2,4)} = 0 \Rightarrow \langle a_{(1,3)}, a_{(2,3)}a_{(2,4)} \rangle = \langle a_{(1,3)}a_{(2,4)}, a_{(2,4)} \rangle = 0$$

But also

$$a_{(2,3)}a_{(2,4)} = \frac{1}{32}(2a_{(2,3)} + 2a_{(2,4)} + a_{(3,4)}) - \frac{135}{2048}u_1$$

So

$$\langle a_{(1,3)}, a_{(2,3)}a_{(2,4)} \rangle = \frac{39}{8192} - \frac{135}{2048}\langle a_{(1,3)}, u_1 \rangle \Rightarrow \langle a_{(1,3)}, u_1 \rangle = \frac{13}{180}.$$

Step 3 - fusion

Recall that V obeys the Majorana fusion rule:

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

So if u and v are already known to be eigenvectors and their product uv is known then we can use this to find new eigenvectors.

Step 3 - fusion

Recall that the spanning set indexed by `coords` is not necessarily a linearly independent set of vectors.

Recall also that for all $a \in A$

$$V = V_1^a \oplus V_0^a \oplus V_{\frac{1}{4}}^a \oplus V_{\frac{1}{32}}^a.$$

Thus, if there exists $v \in V_\mu^a \cap V_\nu^a$ such that $\mu \neq \nu$ then we must have $v = 0$. Such vectors form what we call the nullspace of the algebra.

Step 4 - algebra products

We calculate new algebra products from the following sources:

- if $a \in A$ and $v \in V_\mu^a$ then $av = \mu v$;
- if v is in the nullspace of V then $uv = 0$ for all $u \in V$;
- the resurrection principle.

Each of these leads to a linear equation whose indeterminates are the unknown algebra product values. We use these equations to build a system of linear equations that we solve in order to find new algebra products.

The algorithm

- Step 0 shapes;
- Step 1 setup;
- Step 2 inner products;
- Step 3 fusion;
- Step 4 algebra products.

Loop over steps 2 - 4 until no more algebra products can be found.

If still algebra products remain unknown then extend the spanning set by vectors for the form uv where u and v are in the spanning set of $V^{(2)}$ (the 2-closed part of V) and uv is an unknown product. Again, repeat steps 2 - 4 ...