Worksheet on Finite Geometry in GAP

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1. A *cap* of a projective space is a set $\mathcal{O}$ of points such that no three points of $\mathcal{O}$ are collinear.
   (a) Initialize $\mathrm{PG}(3, q)$, $q = 7$.
   (b) Check that an elliptic quadric $\mathrm{Q}^-(3, 7)$ is indeed a cap of $\mathrm{PG}(3, 7)$.

   Useful commands: `PG`, `EllipticQuadric`, `List`, `Points`, `Lines`, `Number`.
   solution file: `cappg3q.g`.

2. Consider the projective space $\mathrm{PG}(3, q)$, $q = 5$, and its group of projectivities $G = \mathrm{PGL}(4, 5)$. Compute a random element g of G of order $q^2 + q + 1$.
   (a) Compute the orbits of the ponts of $\mathrm{PG}(3, q)$ under the group $\langle g \rangle$.
   (b) Check that most of these orbits span the complete space $\mathrm{PG}(3, q)$.
   (c) Choose one of these orbits that span the full space, and compute the intersection numbers of this orbit with the lines of $\mathrm{PG}(3, q)$.
   (d) Repeat the same steps for $q = 9$ and $q = 17$.
   (e) Could you make a guess about the set of points of such an orbit?

   Useful commands: `ProjectivityGroup`, `Order`, `First`, `FiningOrbits`, `Span`, `Number`.
   solution file: `cappg3q.g`.

3. Consider the classical polar space $\mathrm{H}(5, q^2)$ (the Hermitian variety). This is a rank 3-geometry, it contains points, lines and planes. A *spread* of $\mathrm{H}(5, q^2)$ is a set $\mathcal{S}$ of planes of $\mathrm{H}(5, q^2)$ such that every point of $\mathrm{H}(5, q^2)$ is contained in exactly one plane of $\mathcal{S}$. A *partial spread* is a set $\mathcal{S}$ of planes of $\mathrm{H}(5, q^2)$ such that every point of $\mathrm{H}(5, q^2)$ is contained in at most one plane of $\mathcal{S}$. A partial spread is *maximal* if it cannot be extend to a larger partial spread.
   (a) Initialize the classical polar space $\mathrm{H}(5, q^2)$, $q = 2$.
   (b) Initialize a graph (using the package grape) Γ, with vertex set the set of planes of $\mathrm{H}(5, q^2)$, and two vertices being adjacent if and only if they meet trivially. Note that grape allows to construct graphs using non-trivial symmetry groups.
   (c) Find all maximal partial spreads of $\mathrm{H}(5, q^2)$. Use the grape command `CompleteSubgraphs`.
   (d) Refine your search now by checking the sizes of the found subgraphs, and recomputing, up to isomorphism, all subgraphs of a given size.

   Useful commands: `HermtianPolarSpace`, `Planes`, `Meet`, `CollineationGroup`, `OnProjSubspaces`,
   grape package: `CompleteSubgraphs` `CompleteSubgraphsOfGivenSize`.
   solution file: `mpsh5q2.g`.

4. Consider the point line geometry of points and lines of the parabolic quadric $\mathrm{Q}(4,q)$. This is a classical generalized quadrangle. A *partial ovoid* is a set $\mathcal{S}$ of points such that every line of $\mathrm{Q}(4,q)$ meets $\mathcal{S}$ in at most one point. It is an *ovoid* if every line meets $\mathcal{S}$ in exactly one point. By combinatorial arguments, an ovoid has size exactly q^2+1. A partial ovoid is *maximal* if it cannot be extended to a larger partial ovoid.

   (a) Initialize the parabolic quadric $\mathrm{Q}(4,q)$, $q=5$.

   (b) Initialize a graph (using the package grape) Γ, with vertex set the set of points of $\mathrm{Q}(4,q)$, and two vertices being adjacent if and only if they are not collinear as points of $\mathrm{Q}(4,q)$. Note that grape allows to construct graphs using non-trivial symmetry groups.

   (c) Show that, up to isomorphism, there is only one ovoid of $\mathrm{Q}(4,5)$, and it is an elliptic quadric $\mathrm{Q}^-(3,q)$. Note that this isomorphism check can be done using the grape command(s) in an appropriate way.

   (d) Find the largest maximal partial ovoid of $\mathrm{Q}(4,5)$. Check that it is unique up to isomorphism. Show that the lines not containing a point of this maximal partial ovoid, lie in a hyperplane of $\mathrm{PG}(4,q)$, necessarily meeting $\mathrm{Q}(4,q)$ in a hyperbolic quadric $\mathrm{Q}^+(3,q)$.

   Useful commands: `ParabolicQuadric`, `Points`, `IsCollinear`, `CollineationGroup`, `OnProjSubspaces`, `AmbientGeometry`, grape package: `CompleteSubgraphsOfGivenSize`, `VertexNames`, `Span`.
   solution file: `mpoq4q.g`.

5. The *Klein correspondence*, mapping lines of $\mathrm{PG}(3,q)$ to points of $\mathrm{Q}^+(5,q)$ is quite powerful. Here you will combine this geometry morphism with *field reduction* to describe a spread of a symplectic space.

   (a) Define $q=9$. Initialize $\mathrm{PG}(1,q^2)$ and store all its points in a list.

   (b) Initialize the geometry morphism based on field reduction, mapping points of $\mathrm{PG}(1,q^2)$ to lines of $\mathrm{PG}(3,q)$.

   (c) Use this geometry morphism to construct a spread of $\mathrm{PG}(3,q)$. Check that the set of lines you get is indeed a spread of $\mathrm{PG}(3,q)$.

   (d) Now initialize and use the Klein correspondence to convert the line spread of $\mathrm{PG}(3,q)$ into a set of points of the hyperbolic quadric $\mathrm{Q}^+(5,q)$. Note that the representation of $\mathrm{Q}^+(5,q)$ is not important.

   (e) Show that the point set of $\mathrm{Q}^+(5,q)$ spans a solid S of $\mathrm{PG}(5,q)$, meeting $\mathrm{Q}^+(5,q)$ in a hyperbolic quadric $\mathrm{Q}^+(3,q)$.

   (f) This means that every hyperplane π containing S meets $\mathrm{Q}^+(5,q)$ in a parabolic quadric. Choose any hyperplane π and embed that standard parabolic quadric $\mathrm{Q}(4,q)$ as hyperplane section in $\mathrm{Q}^+(5,q)$. Then compute the preimage of the point set under this embedding.

   (g) Now use that natural duality between $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$ to obtain the line spread of $\mathrm{PG}(3,q)$ as a line spread of $\mathrm{W}(3,q)$, the symplectic generalized quadrangle.

   Useful commands: `NaturalEmbeddingByFieldReduction`, `KleinCorrespondence`, `Range`, `AmbientGeometry`, `ProjectiveDimension`, `Span`,
   `ElementsIncidentWithElementOfIncidenceStructure`,
   `NaturalEmbeddingBySubspace`, `SymplecticSpace`, `NaturalDuality`.
   solution file: `symplectic.g`.

6. We will consider in $\mathrm{PG}(3,q)$, $q = 2^{2e+1}$, a point set of which the coordinates are given as follows.

$$\mathcal{O} = \{(1, st + s^{\sigma+2} + t^{\sigma}, s, t) : s, t \in \mathbb{F}_q\} \cup \{(0,1,0,0)\}\,.$$

   (a) Initalize $\mathrm{PG}(3,q)$, with $q = 8$.
   (b) Use the above list of coordinates to create the set of points $\mathcal{O}$ in $\mathrm{PG}(3,8)$.
   (c) Check that it is a cap.
   (d) Check that every point $p \in \mathcal{O}$ defines a unique *tangent plane*, i.e. a plane meeting $\mathcal{O}$ only in p.
   (e) Create a list with all tangent planes.
   (f) Compute the setwise stabilizer group S of $\mathcal{O}$ in the group of special homographies of $\mathrm{PG}(3,q)$. Compute the order of the group.
   (g) Check which finite simple group you found by computing S.

   Useful commands: `PG`, `VectorSpaceToElement`, `Lines`, `Planes`, `SpecialHomographyGroup`, `FiningSetwiseStabiliser`, `Order`, `StructureDescription`.
   solution file: `titspg3q.g`. The object described here is the famous ovoid of Jacques Tits.

7. We will consider an ovoid in a classical generalized quadrangle. The classical generalized quadrangle will be the point-line geometry on the parabolic quadric $\mathrm{Q}(4,q)$, $q = p^h$, p an odd prime, $h > 1$.

   (a) Let $q = 9$. Initialize the parabolic quadric $\mathrm{Q}(4,q)$ with equation $X_2^2 + X_0X_4 + X_3X_1 = 0$.
   (b) Denote σ the unique non-trivial automorphism of $\mathbb{F}_9$. Let $-k$ be a non-square in $\mathbb{F}_9$. Define on $\mathrm{Q}(4,q)$ the point set with coordinates given as follows

$$\mathcal{O} = \{(1, s, t, ks^{\sigma}, -t^2 - ks^{\sigma+1}) : s, t \in \mathbb{F}_q\} \cup \{(0,0,0,0,1)\}$$

   (c) Check that $\mathcal{O}$ is an *ovoid* of $\mathrm{Q}(4,q)$, i.e. a set of points such that every line of $\mathrm{Q}(4,q)$ meets $\mathcal{O}$ in exactly one point.
   (d) Check that every elliptic quadric contained in $\mathrm{Q}(4,q)$ meets $\mathcal{O}$ in $1 \bmod p$ points.
   (e) Compute the stabilizer group of $\mathcal{O}$ in the special isometry group of $\mathrm{Q}(4,q)$.

   Useful commands: `QuadraticFormByMatrix` (forms package) `PolarSpace`, `AmbientSpace`, `Hyperplanes`, `SpecialIsometryGroup`, `FiningSetwiseStabiliser`.
   solution file: `kantor.g`. The ovoid is an example of non-classical ovoids of $\mathrm{Q}(4,9)$, i.e. an ovoid different from and elliptic quadric (and hence necessarily spanning $\mathrm{PG}(4,q)$).

8. Combine exercises 5 and 6. A famous theorem states that any ovoid $\mathcal{O}$ of $\mathrm{PG}(3,q)$, q even, defines a symplectic generalized quadrangle $\mathcal{S}$ as follows. The points are the points of $\mathrm{PG}(3,q)$, and the lines are the lines of $\mathrm{PG}(3,q)$ tangent to $\mathcal{O}$. Starting from a given ovoid $\mathcal{O}$, one can in principle compute the form. In this exercise, we will reconstruct the symplectic space around the ovoid geometrically using the Klein correspondence. Then we will use the natural duality of $\mathrm{W}(3,q)$ with itself and the geometry isomorphism between symplectic spaces described by different symplectic forms to investigate the two different spreads we get.

   (a) Construct the ovoid $\mathcal{O}$ of exercise (4) in $\mathrm{PG}(3,8)$.
   (b) Compute the lines and planes of $\mathrm{PG}(3,8)$ tangent to $\mathcal{O}$.

(c) Map these tangent lines, respectively planes to points, respectively lines of $\mathrm{Q}^+(5,q)$ using the Klein correspondence.

(d) Check that this set of points (and lines) on $\mathrm{Q}^+(5,q)$ spans a hyperplane of $\mathrm{PG}(5,q)$.

(e) So these points and lines are the points and lines of an embedded parabolic quadric $\mathrm{Q}(4,q)$, embed $\mathrm{Q}(4,q)$ into $q^+(5,q)$ as subspace intersection of $\mathrm{Q}^+(5,q)$.

(f) No use the natural duality between $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$ to convert this particular set of lines of $\mathrm{Q}(4,q)$ to a set of points of $\mathrm{W}(3,q)$. Now check that the obtained point set of $\mathrm{W}(3,q)$ is indeed an ovoid.

Useful commands: `NaturalEmbeddingBySubspace`, `NaturalDuality`, `KleinCorrespondeceExtended`.
solution file: `combination.g`

9. The André-Bruck-Bose representation (ABB) is a representation of *translation planes*. Let S be any spread of $\mathrm{PG}(3,q)$, i.e. a set of lines partitioning the point set. Embed $\mathrm{PG}(3,q)$ as a hyperplane π in $\mathrm{PG}(4,q)$. Now define

   - Points: points of type (i): the points of $\mathrm{PG}(4,q) \setminus \pi$; points of type (ii): the lines of S;
   - Lines: lines of type (a): the planes of $\mathrm{PG}(4,q)$ meeting π in a line of S; line of type (b): the hyperplane π
   - Incidence between Points and Lines is the natural incidence.

(a) Construct (see exercise (5)) a Desarguesian spread of $\mathrm{PG}(3,q)$.

(b) Turn this spread into a non-Desarguesian spread by swapping a regulus. A regulus is determined by three lines l_1, l_2, l_3 as follows. for each point $p \in l_1$, there is a unique line m_p meeting l_1, l_2, l_3 in precisely a point. So the three lines l_1, l_2, l_3 determine $q+1$ lines $m_i : i = 1 \ldots q+1$. This is the opposite regulus R'. The regulus determined by the lines l_1, l_2, l_3 is precisely the opposite regulus determined by any three lines of R'. Clearly, replacing the lines of R by R' yields again a spread of $\mathrm{PG}(3,q)$. Now define the spread $S := (S \setminus R) \cup R'$.

(c) Define the points and lines from the ABB representation with relation to the spread S'.

(d) The incidence is the natural incidence.

(e) Look at the command `GeneralisedPolygonByElements`, you need the compute the stabilizer group of the spread S', and the incidence as an argument for this command is `\*`.

(f) The command `CollineationGroup` is applicable on the generalised polygon you got in (e). However, it will take a lot of time to compute it. If the package `digraph` is available, you can (i) compute the incidence graph using `IncidenceGraph`, (ii) convert this grape object into a digraph object simply by applying `Digraph` on it, (iii) now computing the digraph-automorphism group using `AutomorphismGroup`. Compare the result with the stabilizer group of the non-Desarguesian spread. What is the mathematical conclusion?

solution file: `hall_plane.g`.