Abstract category theory

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Concrete computations in computer algebra

“natural transformation”

Abstract category theory $\Longrightarrow$ Concrete computations in computer algebra

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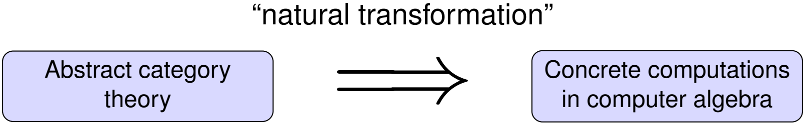
Abstract category theory $\Longrightarrow$ Concrete computations in computer algebra

Categorical abstraction



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Categorical abstraction is a powerful

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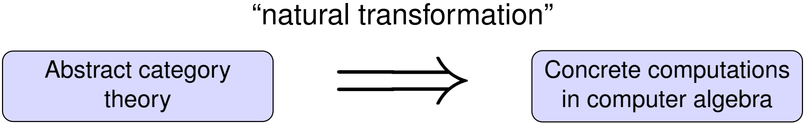
Categorical abstraction is a powerful organizing principle



“natural transformation”

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Concrete computations in computer algebra

Categorical abstraction is a powerful organizing principle and computational tool.

Category theory in computer algebra

Sebastian Posur

November 20, 2018

Outline

Categorical abstraction is a powerful organizing principle and computational tool.

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Categories

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Probably the most important notion in category theory.

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isomorphisms $=$ all elements in G

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isomorphisms $=$ $\mathbb{Q}$-linear bijections

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isomorphisms = invertible matrices

Equivalences

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$$\begin{tikzcd}\mathcal{A} \arrow[r, "F"] & \mathcal{B} \\ A \arrow[r, mapsto] & FA\end{tikzcd}$$

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- $F : \mathrm{Obj}_{\mathcal{A}} \longrightarrow \mathrm{Obj}_{\mathcal{B}}$
- $\mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ (respects id and $\circ$, bijection)
- $\forall B \in \mathcal{B} \quad \exists A \in \mathcal{A} : \quad FA \simeq B$ (essentially surjective)

Equivalences

Let’s compare $\mathrm{mat}_{\mathbb{Q}}$ and $\mathrm{vec}_{\mathbb{Q}}$.

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m & &
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\end{tikzcd}$$

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n \arrow[r, phantom, "\longmapsto"] & & \mathbb{Q}^{1\times n}
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$$\begin{tikzcd}
\mathrm{mat}_{\mathbb{Q}} \arrow[rr, "F"] & & \mathrm{vec}_{\mathbb{Q}} \\
& & \mathbb{Q}^{1\times m} \arrow[d, phantom, sloped, "\simeq"] \\
m \arrow[rru, mapsto] & & B
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- $\forall B \in \mathrm{vec}_{\mathbb{Q}} \quad \exists m \in \mathrm{mat}_{\mathbb{Q}} : \quad \mathbb{Q}^{1\times m} \simeq B \quad m = \dim(B)$

Categorical abstraction

$$\mathrm{mat}_{\mathbb{Q}} \simeq \mathrm{vec}_{\mathbb{Q}}$$

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For a category theorist, these two categories look the same.

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We want to use categories to model **computational contexts** instead of “isolated” objects.

Outline

Categorical abstraction is a powerful organizing principle and computational tool.

1. What is categorical abstraction?
2. How can it be used as an organizing priniciple?
3. Why is it a computational tool?

Computable categories

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A category becomes computable through

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$$\begin{tikzcd} 1 \arrow[r, "\begin{pmatrix} 1 & 2 \end{pmatrix}"'] & 2 \arrow[r, "\begin{pmatrix} 3 \\ 4 \end{pmatrix}"'] & 1 \end{tikzcd}$$

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$\mathrm{vec}_{\mathbb{Q}}$ and $\mathrm{mat}_{\mathbb{Q}}$ are examples of abelian categories.

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$$\ker(\alpha) = \left\{ v \in V \mid \alpha(v) = 0 \right\}$$

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Complete understanding about what is mapped to 0.

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$\dim\left(\ker(a_{ij})\right)$

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$$\dim\big(\ker(a_{ij})\big)$$

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$$\begin{tikzcd}
2 \arrow[dr, "\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}"'] \arrow[drr, bend left, "0"] & & \\
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\end{tikzcd}$$

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$$\begin{tikzcd}
2 \arrow[dr, swap, "\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[drr, bend left, "0"] & & \\
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$$\begin{tikzcd}[ampersand replacement=\&]
2 \arrow[dr, "{\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}}"'] \arrow[drr, bend left, "0"] \& \& \\
\& 3 \arrow[r, "{\begin{pmatrix}1 & -1\\ -1 & 1\\ 1 & -1\end{pmatrix}}"] \& 2 \\
1 \arrow[ur, "{\begin{pmatrix}2 & 1 & -1\end{pmatrix}}"] \& \&
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1 \arrow[ur, "\begin{pmatrix} 2 & 1 & -1 \end{pmatrix}"] \arrow[urr, bend right, "0"'] \arrow[uu, dashed, bend left, "\begin{pmatrix} 2 & -1 \end{pmatrix}"] & &
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$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] & & \\
& A \arrow[r, "\varphi"] & B
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and for every test morphism τ

$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] & & \\
& A \arrow[r, "\varphi"] & B \\
T \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Categorical description of the kernel

Let $\varphi \in \mathrm{Hom}(A, B)$. To fully describe the kernel of φ ...

... one needs an object $\ker\varphi$,
its embedding $\kappa = \mathrm{KernelEmbedding}(\varphi)$,
and for every test morphism τ
a *unique* morphism $\lambda = \mathrm{KernelLift}(\varphi, \tau)$

$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] & & \\
& A \arrow[r, "\varphi"] & B \\
T \arrow[uu, dashed, "\lambda"] \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
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a *unique* morphism $\lambda = \mathrm{KernelLift}(\varphi, \tau)$, such that

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Implementation of the kernel

$\mathrm{mat}_{\mathbb{Q}}$

$$\mathrm{Obj} := \mathbb{N}_0, \qquad \mathrm{Hom}\,(m,n) := \mathbb{Q}^{m\times n}$$

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\mathrm{ker}((a_{ij})_{ij}) & & \\
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Compute

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The language of category theory

The language of category theory

Given a diagram of vector spaces:

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$$\begin{tikzcd}
\color{blue}{\ker} \arrow[r, hook] & A' \arrow[r, blue] \arrow[d, "\alpha"'] & B' \arrow[d] \\
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The language of category theory

The same example in the language of category theory:

$$\begin{tikzcd}
\ker \arrow[r, "\kappa'"] \arrow[d, dashed] & A' \arrow[r] \arrow[d, "\alpha"'] & B' \arrow[d] \\
\ker \arrow[r] & A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

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The same example in the language of category theory:

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{} \arrow[d, dashed] \\
{}
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$$\dashdownarrow \; = \qquad \alpha \circ \kappa'$$

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The same example in the language of category theory:

$$\begin{tikzcd}
\textcolor{blue}{\mathsf{ker}} \arrow[r, "\kappa'"] \arrow[d, dashed] \arrow[dr, red] & A' \arrow[r, blue] \arrow[d, "\alpha"'] & B' \arrow[d] \\
\textcolor{blue}{\mathsf{ker}} \arrow[r] & A \arrow[r, blue, "\textcolor{blue}{\varphi}"] & B
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, dashed] \\ {} \end{tikzcd} = \mathrm{KernelLift}(\textcolor{blue}{\varphi}, \textcolor{red}{\alpha \circ \kappa'})$$

The language of category theory

The same example in the language of category theory:

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\mathrm{ker} \arrow[r, "\kappa'"] \arrow[d, dashed] \arrow[dr] & A' \arrow[r] \arrow[d, "\alpha"'] & B' \arrow[d] \\
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\end{tikzcd}$$

⇣ $= \mathrm{KernelLift}(\varphi, \alpha \circ \kappa')$

This term may be interpreted in other contexts as well.

The language of category theory

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$\mathrm{vec}_{\mathbb{Q}}$

The language of category theory

$\mathrm{vec}_{\mathbb{Q}}$ $\qquad\qquad$ $\mathrm{mat}_{\mathbb{Q}}$

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$$\mathrm{vec}_{\mathbb{Q}} \qquad \simeq \qquad \mathrm{mat}_{\mathbb{Q}}$$

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⋮
KernelObject
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The language of category theory

$$\underbrace{\mathrm{vec}_{\mathbb{Q}} \quad \simeq \quad \mathrm{mat}_{\mathbb{Q}}}_{\text{categorical abstraction}} \quad \circlearrowleft \quad \begin{array}{l} \vdots \\ \text{KernelObject} \\ \text{KernelEmbedding} \\ \text{KernelLift} \\ \vdots \end{array}$$

The language of category theory

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The language of category theory

$$\mathcal{A} \qquad \circlearrowleft \qquad \underbrace{\begin{array}{l} \vdots \\ \text{KernelObject} \\ \text{KernelEmbedding} \\ \text{KernelLift} \\ \vdots \end{array}}_{\text{categorical language}}$$

An introduction to finitely presented modules

Let R be a ring.

Definition

A (left) R-module M is called **finitely presented** if there exist

- $n, m \in \mathbb{N}_0$,
- $r_1, \dots, r_m \in R^{1 \times n}$,
- $M \cong \frac{R^{1 \times n}}{\langle r_1, \dots, r_m \rangle}$

Examples

$$M \cong \frac{R^{1\times n}}{\langle r_1,\ldots,r_m\rangle}$$

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$R = \mathbb{Q}[x, y, z]$

$$\frac{\mathbb{Q}[x, y, z]}{\langle x^2 + y^2 + z^2 - 1\rangle}$$

Examples

$M \cong \frac{R^{1\times n}}{\langle r_1,\ldots,r_m \rangle}$

$R = \mathbb{Q}[x,y,z]$

$$\frac{\mathbb{Q}[x,y,z]}{\langle x^2+y^2+z^2-1 \rangle}$$

$$\frac{\mathbb{Q}[x,y,z]^{1\times 2}}{\left\langle \begin{pmatrix} x & -y \end{pmatrix}, \begin{pmatrix} y & -x \end{pmatrix} \right\rangle}$$

Examples

$M \cong \frac{R^{1\times n}}{\langle r_1,\ldots,r_m\rangle}$

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$$\frac{\mathbb{Q}[x, y, z]^{1\times 6}}{\left\langle \text{Rows of } \begin{pmatrix} 0 & 0 & 0 & 0 & xz & -z^2 \\ 0 & 0 & 0 & 0 & xy & -yz \\ 0 & -x^2z + xyz + xz^2 & y^2z & -xz + yz & x - y & 0 \\ 0 & 0 & 0 & 0 & x^2 & -xz \\ -xy & -x^3 + x^2y + x^2z & xy^2 & -x^2 + xy & 0 & x - y \\ z & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle}$$

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Category of finitely presented modules

Finitely presented *R*-modules form a category

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Finitely presented R-modules form a category

$$\mathrm{mod}_R$$

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with R-linear maps as morphisms.

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Computerfriendly model?

Applying the organizing principle

Goal: create computerfriendly model fpres_R of mod_R.

What we need

1. **Data structures**
   - objects
   - morphisms
2. **Algorithms**
   - composition
   - identities
   - KernelObject
   - $\ldots$

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Data structures: objects

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Idea: a matrix $M \in R^{m\times n}$ can represent the module $\frac{R^{1\times n}}{\langle M\rangle}$.

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Idea: a matrix $M \in R^{m\times n}$ can represent the module $\frac{R^{1\times n}}{\langle M\rangle}$.

Objects

$$\mathrm{Obj}_{\mathrm{fpres}_R} := \biguplus_{m,n\in\mathbb{N}_0} R^{m\times n}$$

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Given: $M \in R^{m \times n}$ and $M' \in R^{m' \times n'}$.

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Given: $M \in R^{m \times n}$ and $M' \in R^{m' \times n'}$.

$$\begin{tikzcd}\frac{R^{1\times n}}{\langle M\rangle} \arrow[rr] & & \frac{R^{1\times n'}}{\langle M'\rangle}\end{tikzcd}$$

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\overline{e_i} &
\end{tikzcd}$$

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\end{tikzcd}$$

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$\mathrm{Hom}_{\mathrm{fpres}_R}(M, M') :=$

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Given: $M \in R^{m \times n}$ and $M' \in R^{m' \times n'}$.

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$A \in R^{n \times n'}$ such that

Data structures: morphisms

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$$\begin{tikzcd}[column sep=huge]
\frac{R^{1 \times n}}{\langle M \rangle} \arrow[r, "A := \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}"] & \frac{R^{1 \times n'}}{\langle M' \rangle} \\
\overline{e_i} \arrow[r, mapsto] & \overline{r_i}
\end{tikzcd}$$

$\mathrm{Hom}_{\mathrm{fpres}_R}(M, M') :=$

$A \in R^{n \times n'}$ such that $\left\{\text{Rows of } M \cdot A\right\} \subseteq \langle M' \rangle$

Data structures: morphisms

Given: $M \in R^{m \times n}$ and $M' \in R^{m' \times n'}$.

$$\begin{tikzcd}
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\end{tikzcd}$$

$\mathrm{Hom}_{\mathrm{fpres}_R}(M, M') :=$

$$A \in R^{n \times n'} \quad \text{such that} \quad \exists X \in R^{m \times m'} : M \cdot A = X \cdot M'$$

Data structures: morphisms

Given: $M \in R^{m \times n}$ and $M' \in R^{m' \times n'}$.

$$\begin{tikzcd}
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$\mathrm{Hom}_{\mathrm{fpres}_R}(M, M') :=$

$$A \in R^{n \times n'} \quad \text{such that} \quad \exists X \in R^{m \times m'} : M \cdot A = X \cdot M'$$

A defines the 0 morphism

Data structures: morphisms

Given: $M \in R^{m\times n}$ and $M' \in R^{m'\times n'}$.

$$\begin{tikzcd}
\frac{R^{1\times n}}{\langle M\rangle} \arrow[r, "A := \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}"] & \frac{R^{1\times n'}}{\langle M'\rangle} \\
\overline{e_i} \arrow[r, mapsto] & \overline{r_i}
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$\mathbb{Q}$	Gauss
$\mathbb{Z}$	Hermite
$\mathbb{Q}[x, y, z]$	Buchberger

Applying the organizing principle

Goal: create computerfriendly model fpres_R of mod_R.

What we need

1. **Data structures**
    - objects ✓
    - morphisms
2. **Algorithms**
    - composition
    - identities
    - KernelEmbedding
    - . . .

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$$\alpha : R \longrightarrow R : r \mapsto \overline{Z} \cdot r$$

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$$\alpha : R \longrightarrow R : r \mapsto \overline{Z} \cdot r$$

has

$$\ker(\alpha) = \langle \overline{X_i} \mid i \in \mathbb{N}\rangle.$$

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Can you prove this theorem?

Outline

Categorical abstraction is a powerful organizing principle and computational tool.

1. What is categorical abstraction?
2. How can it be used as an organizing priniciple?
3. Why is it a computational tool?

Computing the intersection

Let $M_1 \subseteq N$ and $M_2 \subseteq N$ subobjects in an abelian category.

Computing the intersection

Let $M_1 \hookrightarrow N$ and $M_2 \hookrightarrow N$ subobjects in an abelian category.

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$$\begin{tikzcd}
M_1 \arrow[dr, hook, "\iota_1"] & \\
& N \\
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$\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left(\left(M_1, M_2\right), i\right)$, $i = 1, 2$

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Translation to CAP

$\pi_i := \text{ProjectionInFactorOfDirectSum}\left((M_1, M_2), i\right),\ i = 1, 2$

```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
```

$\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2$

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lambda := PostCompose( iota1, pi1 );
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```
gamma := PostCompose( lambda, kappa );
```

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Translation to CAP

```
IntersectionSubobjects := function( iota1, iota2 )



  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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  gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
IntersectionSubobjects := function( iota1, iota2 )

  M1 := Source( iota1 );
  M2 := Source( iota2 );
  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Translation to CAP

```
IntersectionSubobjects := function( iota1, iota2 )
  local M1, M2, pi1, pi2, lambda, phi, kappa, gamma;

  M1 := Source( iota1 );
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  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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  return gamma;
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Computing the intersection

Compute the intersection in $\mathrm{mat}_{\mathbb{Q}}$ of

$$\begin{tikzcd}
M_1 \arrow[r, hook, "\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[d, phantom, "\shortparallel"] & N \arrow[d, phantom, "\shortparallel"] & M_2 \arrow[l, hook', "\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}"'] \arrow[d, phantom, "\shortparallel"] \\
2 & 3 & 2
\end{tikzcd}$$

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$$\begin{tikzcd}
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gap> gamma := IntersectionOfSubobject( iota1, iota2 );
<A morphism in the category of matrices over Q>
```

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\end{tikzcd}$$

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gap> gamma := IntersectionOfSubobject( iota1, iota2 );
<A morphism in the category of matrices over Q>

gap> Display( gamma );
[ [  1,  1,  0 ] ]

A morphism in the category of matrices over Q
```

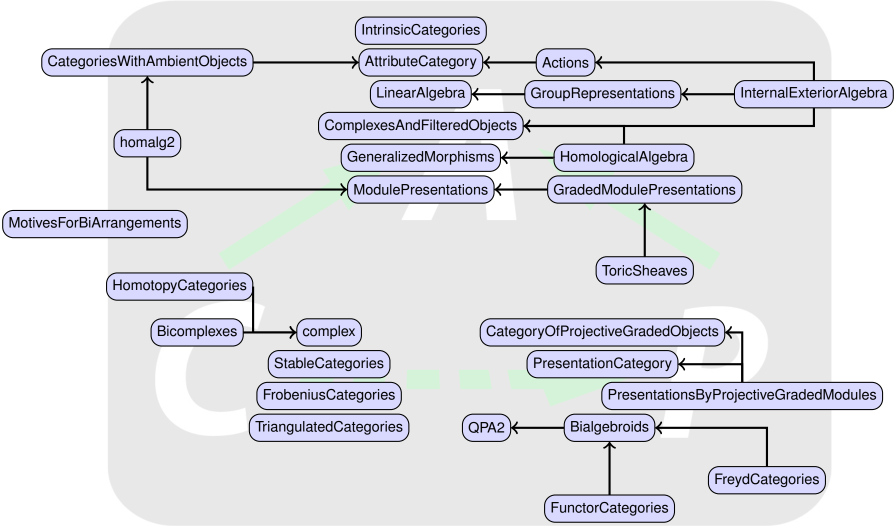
Computing the intersection

The same algorithm can be applied in fpres_R

Computing the intersection

The same algorithm can be applied in fpres_R (your turn).

CAP packages



IntrinsicCategories

CategoriesWithAmbientObjects

AttributeCategory

Actions

LinearAlgebra

GroupRepresentations

InternalExteriorAlgebra

ComplexesAndFilteredObjects

homalg2

GeneralizedMorphisms

HomologicalAlgebra

ModulePresentations

GradedModulePresentations

MotivesForBiArrangements

ToricSheaves

HomotopyCategories

Bicomplexes

complex

CategoryOfProjectiveGradedObjects

StableCategories

PresentationCategory

FrobeniusCategories

PresentationsByProjectiveGradedModules

TriangulatedCategories

QPA2

Bialgebroids

FreydCategories

FunctorCategories

Categorical abstraction is a powerful organizing principle and computational tool.